ARTISTIC PROOFS: A KANTIAN APPROACH TO AESTHETICS IN MATHEMATICS

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This paper explores the nature of mathematical beauty from a Kantian perspective. According to Kant's *Critique of the Power of Judgment*, satisfaction in beauty is subjective and non-conceptual, yet a proof can be beautiful even though it relies on concepts. I propose that, much like art creation, the formulation and study of a complex demonstration involves multiple and progressive interactions between the freely original imagination and taste (that is, the aesthetic power of judgement). Such a proof is artistic insofar as it is guided by beauty, namely, the mere feeling about the imagination's free lawfulness. The beauty in a proof's process and the perfection in its completion together facilitate a transition from subjective to objective purposiveness, a transition that Kant himself does not address in the third *Critique*.

Many would call mathematical principles, such as the Law of Sines, elegant or beautiful. Mathematicians, in particular, underscore the aesthetic value of their works. For instance, Bertrand Russell asserts that 'mathematics, rightly viewed, possesses not only truth, but supreme beauty'.[1] And, according to G. H. Hardy, 'beauty is the first test: there is no permanent place in the world for ugly mathematics.'[2] However, some philosophers remain sceptical about so-called mathematical beauty. Nick Zangwill suggests we distinguish '*intellectual* pleasures' in proofs and theories from 'aesthetic pleasures'.[3] Cain Todd also claims that the alleged 'aesthetic' judgements in mathematics are 'too closely correlated' with logical and epistemic considerations to warrant unequivocally being called aesthetic.[4]

In his *Critique of the Power of Judgment*, Kant characterizes judgements of taste as 'aesthetic' and only determinable by the feeling of pleasure and

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1 Bertrand Russell, *Mysticism and Logic, and Other Essays* (London: Longmans, 1919), 60.

2 G. H. Hardy, *A Mathematician's Apology* (1940; University of Alberta Mathematical Sciences Society, 2005), https://www.math.ualberta.ca/mss/misc/A%20Mathematician%27s%20Apology.pdf, 14.

3 Nick Zangwill, *The Metaphysics of Beauty* (Ithaca, NY: Cornell University Press, 2001), 140.

4 Cain Todd, 'Fitting Feelings and Elegant Proofs: On the Psychology of Aesthetic Evaluation in Mathematics', *Philosophia Mathematica* 26 (2018): 216.

displeasure.[5] In judging a beautiful form, we find in our cognitive faculties a 'lawfulness without law', insofar as their operation is harmonious and yet free from conceptual determination (*CJ*, AA 5:240). On this basis, Kant rejects the notion of 'intellectual beauty' and regards a shape or a number's property of usefulness as a 'relative perfection' (*CJ*, AA 5:366).[6] By distinguishing beauty from perfection, Kant aims to free aesthetic experience from the demands of morality and objective cognition.

Nevertheless, in § 62 of the third *Critique*, Kant also states that 'it would be better to be able to call a *demonstration* of such properties beautiful', even though the basis of the satisfaction 'lies in concepts' (*CJ*, AA 5:366),[7] which seems to contradict his own precept of 'aesthetic' judgements. Kant himself does not elaborate on this point, for the main concern of § 62 is a kind of formal, objective purposiveness in mathematics. And so, this paper does not aim to interpret Kant's particular statement of beautiful demonstrations but rather approaches the problem of mathematical beauty from a Kantian perspective. In turn, the discussion will also shed light on Kant's theory of taste and artistic creation. Admittedly, given Kant's explicit dismissal of 'intellectual beauty', this approach focuses on rudimentary mathematics, especially Euclidian geometry, as reflected in Kant's examples and which is familiar to common people, but the implications thereof may be applicable to issues in contemporary higher mathematics.

I propose that, much like the creation of a beautiful artwork, the formulation of a complex demonstration involves multiple and progressive interactions between the freely original imagination and taste. Without knowing the whole series of subsidiary steps, we start a mathematical proof by trying a certain initial step on account of its compatibility with some as-yet indeterminate concept. In presenting this step, the imagination plays harmoniously with the understanding without any conceptual accordance. This free lawfulness of

5 Immanuel Kant, *Critique of the Power of Judgment*, trans. Paul Guyer and Eric Matthews (Cambridge: Cambridge University Press, 2001), AA 5:203. Hereafter: *CJ*.

6 Kant sometimes distinguishes between the 'utility' and the 'perfection' of an object, that is, between its 'external' and 'internal' objective purposiveness (*CJ*, AA 5:226). However, this paper will follow his terminology in *CJ*, AA 5:366, and employ 'usefulness' and 'perfection' interchangeably.

7 Wenzel argues that 'Kant's theory of "free play" and "purposiveness" is useful in seeing and explaining (contrary to Kant's own claims) how taste matters in mathematical discoveries'. Christian Helmut Wenzel, 'Mathematics and Aesthetics in Kantian Perspective', in *I, Mathematician*, vol. 2, *Further Introspections on the Mathematical Life*, ed. Peter Casazza, Steven G. Krantz, and Randi D. Ruden (Bedford, MA: Consortium of Mathematics, 2016), 103. However, is Wenzel's thought indeed contrary to all of Kant's claims? There is a tension between Kant's general theory of taste and his argument in *CJ*, AA 5:366, which Wenzel does not seem to take into account.

the imagination grounds the purely aesthetic satisfaction in a judgement of taste. Such a proof is 'artistic', even though it eventually justifies a mathematical principle and rests on conceptual accordance. The beauty in a proof's process and the perfection in its completion together facilitate a transition from subjective purposiveness to objective purposiveness in the third *Critique*, a transition that Kant himself fails to address.

This paper comprises four sections. Section I presents an analysis of Kant's distinction between beauty and perfection. With reference to Kant's theory of artistic creation, Section II investigates the interaction between the freely original imagination and the power of judgement in doing mathematics. Further, Section III proposes their progressive interactions, whereby the aesthetic power of judgement (that is, taste) brings free lawfulness to the imagination.

I

In the *Critique of the Power of Judgment*, Kant distinguishes between two types of satisfaction as follows.[8]

The first, satisfaction in perfection, lies in an object's accordance with the concept of what it ought to be (*CJ*, AA 5:226–27). As Kant writes in the second Introduction to the *Critique*, we have this cognitive aim to obtain a coherent order of nature in its particular laws, such that we feel 'noticeable pleasure, often indeed admiration' in comprehending 'empirical heterogeneous laws of nature under a principle' (*CJ*, AA 5:187). Interestingly, in § 62 of the *Critique*, where Kant discusses mathematical theorems and properties of shapes and numbers, he uses exactly the same expression and asserts an 'admiration' for the 'unification of heterogeneous rules […] in one principle' (*CJ*, AA 5:365).

In Kant's example, we can construct a triangle with a given baseline and a given angle opposite to it in infinitely many ways, yet 'the circle comprehends them all, as the geometrical locus for all triangles that satisfy this condition' (*CJ*, AA 5: 362).[9] Kant seems to refer to a consequence of the Law of Sines. Figure 1 below shows that the circle O, which comprehends the given baseline AB and the given angle ∠C, also comprehends all triangles with the same

8 Discussions of this distinction are scattered in many places in the third *Critique*, for example, §§ 3, 15, 34, 62. Kant actually proposes a trichotomy of satisfaction, the third type being the satisfaction in the agreeable (for example, the pleasure we take in fresh air), an investigation of which surpasses the scope of this paper.

9 Kant suggests several examples of geometrical theorems. For an extensive discussion, see Courtney David Fugate, '"With a Philosophical Eye": The Role of Mathematical Beauty in Kant's Intellectual Development', in *Kant: Studies on Mathematics in the Critical Philosophy*, ed. Emily Carson and Lisa Shabel (Abingdon: Routledge, 2016), 241–49.

baseline and opposite angles, such as ΔABD and ΔABE, whereas ∠C, ∠D, and ∠E must be equivalent in degree. Figure 2 shows one of the many ways to prove this: we can draw subsidiary lines OA, OB, and OC, construct three isosceles triangles and thereby demonstrate that the degree of ∠C is a constant wherever point C falls on the circle.

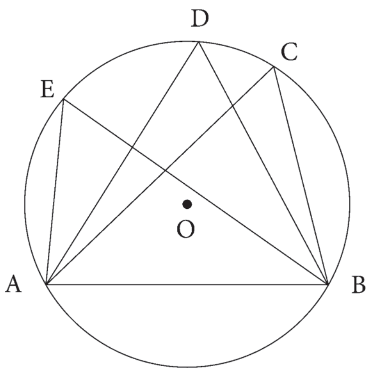

Fig. 1

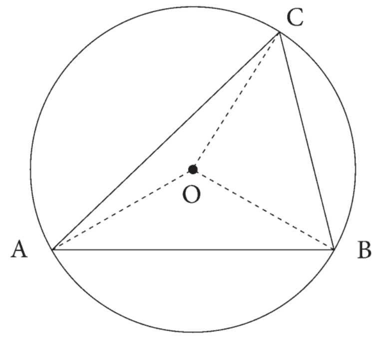

Fig. 2

The unification of the heterogeneous shapes (that is, the various angles) in one principle (that is, the concept of the circle) accords with our cognitive pursuit of systematicity and thus brings about noticeable pleasure or even admiration in the perfection of these shapes.

The second type of satisfaction Kant discusses is satisfaction in beauty. A judgement of taste is 'aesthetic' or 'subjective', which means its sole determining ground is 'the feeling of pleasure and displeasure' (*CJ*, AA 20:224). While a judgement (*Urteil*) of taste is a verdict about the relation between a form and my feeling, our mental faculty for perceiving this feeling is taste, which is a use of the aesthetic power of judgement (*Urteilskraft*).[10] The use of taste, then, is the mental operation of judging (*Beurteilung*) something by mere pleasure or displeasure.[11]

For Kant, the power of judgement in general is 'the faculty of *subsuming* under rules'.[12] Since there cannot be another objective rule for determining

10 the other use of the aesthetic power of judgement is the judging of sublimity (*CJ*, AA 5:244–78), which is irrelevant to our concern here. In the remaining text I shall identify the aesthetic power of judgement with taste.

11 As Vandenabeele points out, we should distinguish between 'the act of judging or contemplating the object [*Beurtheilung des Gegenstandes*]' and 'the judgment of taste [*Geschmacksurtheil*] as such'. Bart Vandenabeele, 'Beauty, Disinterested Pleasure, and Universal Communicability: Kant's Response to Burke', *Kant-Studien* 103 (2012): 225.

12 Immanuel Kant, *Critique of Pure Reason*, trans. Paul Guyer and Allen W. Wood (Cambridge: Cambridge University Press, 1998), AA A132/B171.

whether something stands under a rule, Kant calls the power of judgement a 'special talent' which cannot be taught, and which decides whether a manifold of particular elements or objects stand under a concept.[13] Even a determination of perfection requires the power of judgement, where this faculty judges a manifold's accordance with a determinate concept through satisfaction in perfection.

On this basis, Kant proposes that, in judging the beautiful, the aesthetic power of judgement (that is, taste) judges a manifold's accordance with some '*indeterminate* concept' (*CJ*, AA 5:341). For Kant, the imagination is the faculty for the 'composition of the manifold of intuition', while the understanding is the faculty for the 'unity of the concept that unifies the representations' (*CJ*, AA 5:217). In cognition, the imagination represents to our mind a manifold of items such as lindens and willows, while the understanding is to comprehend them under the concept 'tree' and to determine the imagination's activity accordingly. In judging the beautiful, however, the two cognitive faculties remain in a subjective 'free play' (*CJ*, AA 5:217).[14] A beautiful form can be described in all sorts of words, but its beauty is ineffable and cannot be grasped by concepts.

Kant's distinction between beauty and perfection establishes the freedom of the aesthetic from the demands of morality and objective cognition. In daily language we may call a pattern 'beautiful' on account of its correspondence to the golden ratio and the pleasure thereof, but how could we differentiate such pleasure from what we feel in judging a well-organized shelf or a precisely sliced cake? From a Kantian perspective, these are all instances of perfection or 'regularity' *(Regelmäßigkeit)*, that is, a thing's conformity with its rule (*Regel*), or concept (*CJ*, AA 5:242). By contrast, the creation and appreciation of beauty exists on a non-conceptual basis which grounds the mere feeling of pleasure. This enables us to understand Kant's dismissal of the notion of 'intellectual beauty': as a consequence of the Law of Sines, the aforementioned useful property of a circle for constructing certain triangles cannot be called beautiful, because we judge them intellectually 'in accordance with concepts' (*CJ*, AA 5:366).

Nevertheless, Kant states that 'it would be better to be able to call a *demonstration* of such properties beautiful', because 'the satisfaction, although its ground lies in concepts, is subjective, whereas perfection is

13 Ibid., AA A133/B172.

14 For a detailed discussion on the free play, see Weijia Wang, 'Three Necessities in Kant's Theory of Taste: Necessary Universality, Necessary Judgment, and Necessary Free Harmony', *International Philosophical Quarterly* 58 (2018): 255–73.

accompanied with an objective satisfaction' (*CJ*, AA 5:366).[15] For Kant, 'demonstration' can be synonymous with schematic hypotyposis, whereby the imagination presents a concept by intuition (*CJ*, AA 5:342, 352), but a demonstration, thus understood, must be immediately conceptual and non-beautiful. It is noteworthy that what Kant characterizes as beautiful in § 62 is specifically a demonstration of 'such properties' (*CJ*, AA 5:366), such as the circle's usefulness for constructing certain triangles, and we demonstrate this property by proving the Law of Sines. Therefore, we should read Kant's statement here in light of his definition of 'demonstration' elsewhere as a 'mathematical proof'.[16] Either way, a demonstration always keeps a determinate concept in view, which normally entails its objectivity.[17] And yet, Kant calls the satisfaction 'subjective', which means it does not derive from any conceptual unification or provision of understanding.

Kant does not elaborate on how exactly a demonstration can be beautiful. Nevertheless, as I shall show, it is very rewarding to inquire into the nature of mathematical beauty, especially that in Euclidian geometry, from a Kantian perspective. In the next section, I shall argue that the power of judgement brings lawfulness to the imagination's free originality in artistic creation.

II

Kant defines 'genius' as the 'talent (natural gift) that gives the rule to art' (*CJ*, AA 5:306). Art presupposes a rule, which cannot be couched in a formula, and which must be abstracted from the deed (*CJ*, AA 5:309). Genius displays itself not so much in presenting a determinate concept as in expressing an aesthetic idea, namely, a 'representation of the imagination that occasions

15 As Wenzel notices, in his *Lectures on Anthropology*, the pre-*Critique* Kant once claims that 'mathematical figures are not beautiful, but demonstrations in geometry can be beautiful due to their shortness, their completeness, their natural light, and their suitability for an easier understanding'. Immanuel Kant, *Vorlesung Collins (Wintersemester 1772/1773)*, in *Gesammelte Schriften* (Berlin: de Gruyter, 1997), AA 25:183 (translated by Wenzel). This may reflect a kind of ambivalence in Kant's thoughts. Christian Helmut Wenzel, 'Beauty, Genius, and Mathematics: Why Did Kant Change His Mind?', *History of Philosophy Quarterly* 18 (2001): 426; 'Mathematics and Aesthetics', 99.

16 Immanuel Kant, *Lectures on Logic*, trans. Michael Young (Cambridge: Cambridge University Press, 1992), AA 9:71, 24:234, 893.

17 In the third *Critique*, Kant usually identifies 'objective' with 'conceptual' and associates both with 'perfection' (*CJ*, AA: 226–29, § 15). One might argue that Kant refers a beautiful demonstration not to the process of a proof but to its product, such as a regular figure which demonstrates or presents a geometrical concept; in this case, we aesthetically play with the figure by bracketing its concept and beholding it in a subjective way. Interesting as it is, I would not consider this a faithful interpretation of Kant's text; for a demonstration, qua schematic hypotyposis or a mathematical proof, necessarily involves consideration of a concept and cannot abstract from it.

much thinking though without it being possible for any determinate thought, i.e., *concept*, to be adequate to it' (*CJ*, AA 5:314). As components of genius, our 'imagination' produces such an aesthetic idea and our 'spirit' finds its expression in an 'aesthetic attribute', which 'gives the imagination cause to spread itself over a multitude of related representations' (*CJ*, AA 5:313–15, comp. 5:320).

In Kant's example, an artist's imagination produces the aesthetic idea of Jupiter's heavenly powerfulness, which contains infinite intuitions or 'much thinking'. Further, the artist's spirit conveys this otherwise ineffable aesthetic idea through an 'aesthetic attribute' that is Jupiter's eagle. The eagle, with the lightning in its claws, stimulates the judging subject's imagination to re-create the aesthetic idea of Jupiter's powerfulness; and so, the artist communicates his idea to the audience.

For Kant, genius features 'originality' (*CJ*, AA 5:307), which means the imagination's 'freedom from all guidance by rules' in producing aesthetic ideas (*CJ*, AA 5:317). Meanwhile, Kant characterizes the mental state in artistic creation as a 'free correspondence of the imagination to the lawfulness of the understanding' (*CJ*, AA 5:317). The key question is: how should we reconcile these two types of freedom? I associate them with two correlated, yet clearly distinct, respects of 'productive imagination' that Kant examines in the third *Critique*.

The first respect refers to the imagination's free originality. In *Anthropology from a Pragmatic Point of View*, Kant defines the 'productive' imagination as 'a faculty of the original presentation of the object (*exhibitio originaria*), which thus precedes experience'.[18] Similarly, in the third *Critique*, Kant describes the imagination in artistic creation as 'a productive cognitive faculty', which is 'very powerful in creating, as it were, another nature, out of the material which the real one gives it' (*CJ*, AA 5:314). On both occasions, Kant calls the imagination 'productive' in view of its originality or creativity.

For Kant, even when the imagination reproduces certain empirical materials, this faculty may still produce a manifold of intuitions insofar as it arbitrarily associates the materials in a fashion which is not derivative from, and thus not preceded by, experience.[19] The two functions are often intertwined. For instance, although a musician requires empirical acoustic elements to compose, it is his original imagination that, in reproducing these given sensations, associates them in various original manners.

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18 Immanuel Kant, *Anthropology from a Pragmatic Point of View*, in *Anthropology, History, and Education*, trans. Robert B. Louden (Cambridge: Cambridge University Press, 2007), AA 7:167.

19 Ibid., AA 7:168.

We consider the imagination original when it brings to mind some shape following an a priori principle, for such a presentation does not rely on experience. On the other hand, we consider the imagination freely original when it presents an intuition even without following certain a priori principles. This is consistent with daily language. We would call calligraphy 'original' or 'creative' only when the writing does not entirely follow pure, geometrical laws such as symmetry, cursiveness, and so on. Calligraphy does not necessarily violate or completely abandon such laws, but its originality consists in how it surpasses their limits.

The second respect of the 'productive imagination' refers to the imagination's free lawfulness. In the first *Critique*, Kant calls the imagination 'productive' insofar as its synthesis is an 'exercise of spontaneity', which determines the form of sense a priori 'in accordance with the unity of apperception' and 'in accordance with the categories'.[20] For Kant, the 'unity of apperception', in relation to the synthesis of the imagination, is the understanding,[21] while the 'categories' are nothing other than the pure concepts of the understanding.[22] In this context, the imagination's operation is 'productive' in terms of its accordance with the understanding's pure concepts or laws. By contrast, the imagination is 'reproductive' when determined by empirical laws.

This sheds light on Kant's statement in the third *Critique* that the imagination, in its 'free lawfulness' (*freie Gesetzmäßigkeit*), is taken 'not as reproductive, as subjected to the laws of association, but as productive and self-active (as the authoress of voluntary forms of possible intuitions)' (*CJ*, AA 5:240). In line with the first *Critique*, Kant hereby describes the imagination as 'productive' in view of its lawfulness and its freedom from empirical laws. Nevertheless, as Kant further points out in the same paragraph, in representing a beautiful form, the imagination enjoys even more freedom, as its lawfulness is now 'without law' and its correspondence to the understanding becomes 'subjective' or non-conceptual (*CJ*, AA 5:241). Hence, the imagination accords with the understanding's lawfulness only in general and not with any determinate laws.[23]

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20 Kant, *Critique of Pure Reason*, AA B151–52.

21 Ibid., AA A119.

22 Ibid., AA A79/B105–A80/B105.

23 Following Bernard's translation of '*freie Gesetzmäßigkeit*' as 'free conformity to law', Makkreel asserts that the imagination hereby 'conforms to laws that are still the laws of the understanding'. Rudolf A. Makkreel, *Imagination and Interpretation in Kant: The Hermeneutical Import of the 'Critique of Judgment'* (Chicago: University of Chicago Press, 1990), 47. Similarly, following Bernard's translation of '*gesetzmäßig*' (*CJ*, AA 5:241) as 'conformed to law' instead of 'lawful', Zammito declares that 'the imagination is operating in accordance with law without yet being aware of it

This leads to Kant's assertion that the imagination displays a kind of freedom when 'it schematizes without a concept' (*CJ*, AA 5:287). For Kant, to schematize a concept, the imagination generates the 'representation of a general procedure' for providing the concept with an image and thereby prepares itself for exhibiting the concept in various representations.[24] Now that the understanding aims at unifying representations through a concept, the imagination, to schematize without any (determinate) concept, is disposed to exhibit some concept in general and to accord with the understanding's lawfulness only in general.[25]

In its free lawfulness, the imagination is possibly 'bound to a determinate form' and 'to this extent has no free play (as in invention)' (*CJ*, AA 5:240). When we judge a beautiful rose, our imagination must represent the rose's given form, such that its operation is only free from conceptual accordance but not free from experience. This explains why artistic and mathematical creation, qua original creation, presupposes the imagination's free originality.

Now we can draw two important differences between the two respects of the 'productive imagination'.

Firstly, the imagination's free, original production is not necessarily lawful. As Kant puts it, 'there can also be original nonsense' (*CJ*, AA 5:307). It is one thing that the imagination produces some form, but it is quite another that the imagination, in presenting this form, conforms to the understanding's lawfulness. Moreover, the judging of this conformity (if any) requires the power of judgement, a faculty that is not involved in the original production itself.

For Kant, while the richness of the imagination produces 'nothing but nonsense' through its 'lawless freedom', taste is 'the discipline (or corrective) of genius'; and so, to ask whether genius or taste is more essential in creation of beautiful art is to ask 'whether imagination or the power of judgment counts for more' (*CJ*, AA 5:319). It remains as controversial whether taste should be internal

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and expressly observing it'. John H. Zammito, *The Genesis of Kant's 'Critique of Judgment'* (Chicago: University of Chicago Press, 1992), 114. I disagree with both commentators, for Kant explicitly states that the mental state is 'a lawfulness without law' (*CJ*, AA 5:241), that is, a lawfulness free from empirical as well as pure laws. In my view, Bernard's translations are not erroneous in themselves but do not square with the context. Comp. Immanuel Kant, *Critique of Judgment*, trans. J. H. Bernard (London: Macmillan, 1914), 96.

24 Kant, *Critique of Pure Reason*, AA A140/B179–A140/B180.

25 I believe this is the mental disposition which Henrich calls 'the conditions of a possible conceptualization in general'. Dieter Henrich, *Aesthetic Judgment and the Moral Image of the World: Studies in Kant* (Stanford, CA: Stanford University Press, 1992), 49. For a detailed discussion on the imagination's 'free lawfulness', see Hannah Ginsborg, 'Lawfulness without a Law: Kant on the Free Play of Imagination and Understanding', *Philosophical Topics* 25 (1997): 37–81.

or external to genius,[26] but we can safely interpret taste as external to the freely original imagination – their relation will become our primary concern in Section III.

Secondly, the imagination can be free from certain conceptual guidance in its original creativity; by contrast, it is free from certain conceptual accordance in its free lawfulness. When the imagination arbitrarily brings to mind an aesthetic idea – for instance, countless sorts of forms without aiming at any geometrical concept – a freely originated form may nevertheless contingently fall under the concept of 'circle', such that the imagination displays lawfulness in presenting this form; and yet, this lawfulness is not free but is confined by the geometrical concept. On the other hand, a form may contain 'precisely such a composition of the manifold as the imagination would design in harmony with the *lawfulness of the understanding* in general if it were left free by itself' (*CJ*, AA 5:240–41). In this case, whether the form is original or given, the aesthetic power of judgement judges the imagination's free lawfulness by a mere feeling of pleasure. To summarize, it is one thing to freely and originally produce an aesthetic idea and its expression in an aesthetic attribute or a form, but it is quite another that certain features of this form are not determinable under any concept; only in view of such features do we judge a form to be beautiful.[27]

According to Kant, it is with regard to the imagination that an artwork deserves to be called 'inspired', that is, 'rich and original in ideas', but more importantly, it is with regard to the power of judgement that an artwork deserves to be called 'beautiful', insofar as the imagination in its freedom

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[26] Guyer argues that taste is 'internal' to genius, but he admits this reading's difficulty in answering why Kant contrasts taste to genius as the latter's discipline or corrective. Paul Guyer, 'Genius and Taste: A Response to Joseph Cannon, "The Moral Value of Artistic Beauty in Kant"', *Kantian Review* 16 (2011): 130, 132. Meanwhile, Cannon holds that genius exhibits aesthetic ideas 'in exemplary (but not yet beautiful) works of art', such that 'genius *involves* the power of judgment' but not 'a free *exercise* of judgment'. Joseph Cannon, 'Reply to Paul Guyer', *Kantian Review* 16 (2011): 136. However, I find Cannon's premise implausible, for Kant does ascribe to genius the disposition of 'free correspondence of the imagination to the lawfulness of the understanding', namely, the disposition in our judging of beauty (*CJ*, AA 5:317–18). As I see it, Guyer's and Cannon's readings correspond respectively to what Allison calls the 'thick' and the 'thin' conceptions of genius and reflect ambivalence in Kant's thoughts. See Henry E. Allison, *Kant's Theory of Taste: A Reading of the 'Critique of Aesthetic Judgment'* (Cambridge: Cambridge University Press, 2001), 301.

[27] Cassirer ascribes both cases to the so-called 'aesthetic imagination', which is from both empirical and pure laws. As I see it, the notion of 'aesthetic imagination', not incorrect in itself, obscures the crucial difference between the imagination's two types of freedom. H. W. Cassirer, *A Commentary on Kant's 'Critique of Judgment'* (London: Methuen, 1970), 217, 280–81.

nevertheless conforms to the understanding (*CJ*, AA 5:319). Indeed, Kant specifies the work of genius as 'beautiful art' (*CJ*, AA 5:308). An aesthetic attribute represents an idea beautifully. While the power of judgement in general imposes lawfulness on the freely original imagination, in the creation of a 'beautiful art', the aesthetic power of judgement disciplines the imagination without concepts and brings it into free lawfulness.

By analogy, I propose that, in formulation of a mathematical demonstration, the freely original imagination also interacts with the power of judgement. On the one hand, the imagination can introduce countless intermediate steps (that is, an aesthetic idea) without entirely relying on conceptual guidance. On the other hand, I argue that the power of judgement 'sifts' the multitude of possible steps so as to select the one step that is conducive to the eventual proof. Mathematicians, much like artists, often arbitrarily consider certain shapes or numbers and associate them in various ways that are purely subjective. Hence, the two differ not so much in their imaginations' free originality as in its free lawfulness.

It must be emphasized that an artist aims to express his or her aesthetic ideas in aesthetic attributes, which convey aesthetic ideas through 'the free correspondence of the imagination to the lawfulness of the understanding', that is, beautifully. By contrast, mathematicians communicate ideas by 'logical attributes', which 'represent what lies in our concepts' (*CJ*, AA 5:314). For instance, ultimately, we judge a subsidiary line to be conducive to the proof of the Law of Sines in terms of its compatibility with the concept of this proof. The line is evidently a logical attribute which belongs to the conceptual unity. In this case, the power of judgement disciplines the imagination's creativity according to a determinate law, such that the imagination displays free originality but not free lawfulness.

Given the distinction between aesthetic and logical attributes, how can a demonstration be beautiful? It is tempting to appeal to Kant's notion of adherent or dependent beauty (*anhängende Schönheit*), namely, beauty that presupposes an object's perfection according to its concept (*CJ*, AA 5:229). Insofar as a beautiful artwork is 'a beautiful representation of a thing' (*CJ*, AA 5:311), its beauty presupposes the concept of the 'thing' and is adherent. By analogy, it seems that a mathematical proof can be adherently beautiful in constructing a conceptual unity, much in the same way that a portrait can be beautiful in depicting a certain person.

Nevertheless, as Zangwill points out, an object's adherent beauty can 'come apart' from the object's functionality or perfection; by contrast, a dysfunctional proof cannot be beautiful, such that its appreciation is not genuinely aesthetic.[28] Indeed, a portrait's beauty consists not in how perfectly it represents a person but in the particular way it accomplishes this task, a way by which the artist beautifully conveys an aesthetic idea, while a completed proof is evaluated solely in terms of its perfection. Rieger disagrees with Zangwill on the grounds that '*some* beauty may depend on actual success in fulfilling the function'.[29] But as I see it, while a work's adherent beauty does depend on its perfection, the latter is a necessary but insufficient condition. As Allison nicely puts it, 'a judgement of adherent beauty is not *purely* a judgement of taste, though the taste component within the complex evaluation itself remains *pure*'.[30] We would judge an artwork (qua artwork) to be beautiful only if it is 'academically correct' (*CJ*, AA 5:310), but we never judge its beauty in terms of its correctness. For Kant, although adherent beauty presupposes conceptual accordance, 'perfection does not gain by beauty, nor does beauty gain by perfection' (*CJ*, AA 5:231). By contrast, we would not retain a certain step in a demonstration unless it is functional and compatible with the latter's concept. Hence, the satisfaction entirely consists in the step's functionality.

Rieger argues that there is more to a proof's evaluation than its 'simple effectiveness', for otherwise 'any two correct proofs of the same theorem would be on a par'.[31] Indeed, we may prefer one proof to another even when they both demonstrate the same theorem, but is this preference aesthetic? For instance, there exist two proofs for the irrationality of $\sqrt{2}$: the first makes use of odd and even numbers, the second prime numbers. Cellucci claims that the second proof is 'beautiful' because 'it provides understanding' by showing the real reason 'why $\sqrt{2}$ cannot be a fraction'.[32] But, from a Kantian perspective, I would contend that we favour the second proof because it serves to demonstrate the irrationality of square roots of *all* prime numbers, which gives the real reason for the irrationality of $\sqrt{2}$ (that is, a particular case); by contrast, the first proof applies exclusively to a particular case. In this regard, the second proof is not more beautiful but more useful or better, as it justifies

28 Zangwill, *Metaphysics of Beauty*, 141–42.

29 Adam Rieger, 'The Beautiful Art of Mathematics', *Philosophia Mathematica* 26 (2018): 242.

30 Allison, *Kant's Theory of Taste*, 290.

31 Rieger, 'Beautiful Art', 243.

32 Carlo Cellucci, 'Mathematical Beauty, Understanding, and Discovery', *Foundation of Science* 20 (2015): 348–49. In the same vein, Davis and Hersh claim that the second demonstration 'exhibits a higher degree of aesthetic delight' because it 'seems to reveal the heart of the matter'. Philip J. Davis and Reuben Hersh, *The Mathematical Experience* (New York: Houghton Mifflin, 1981), 299. Hardy also regards this as an example of a 'beautiful theorem' (*Mathematician's Apology*, 19–20). It is with such 'beauty' in view, I believe, that Hardy famously states 'I am interested in mathematics only as a creative art' (ibid., 30).

the unification of more particular numbers under a more general concept or theorem. Similarly, Euclid's proof of the infinitude of prime numbers, elegant and simplistic as it is, should be considered an example of perfection rather than beauty.

Hence, the result or eventual success of a demonstration only brings about a satisfaction in perfection. Thus far I have shown that, much like in the creation of artworks, the power of judgement imposes lawfulness on the freely original imagination in the formulation of proofs. The question remains as to how the aesthetic power of judgement or taste brings free lawfulness to the imagination in doing mathematics. The next section will show how a proof can be beautiful in its process.

III

I propose that the processes of formulating and studying a complex proof require multiple steps and progressive interactions between the freely original imagination and taste. My account is consistent with Kant's aesthetics but not spelled out by Kant himself. The mental operation in question is again analogous to that of artistic creation.

I suggest that, for the artist who attempts to convey Jupiter's powerfulness in an eagle, he or she would sketch a beautiful outline of the eagle using his or her taste, on the basis of which his or her freely original imagination further generates a great multitude of intuitions about the details. As Merleau-Ponty poetically describes, in painting, Matisse's brush may 'meditate […] to try ten possible movements, dance in front of the canvas, brush it lightly several times, and crash down finally like a lightning stroke upon the one line necessary'.[33] All the possible intuitions can become a part of the eagle, but not all of them can fit into a beautiful form and express the aesthetic idea. Once again, it requires taste to single out the subjectively suitable intuition, say, two curves paired with a triangle, while no objective principle can explain why the two match each other. In other words, taste brings free lawfulness to the imagination's free originality. Only then does the imagination further generate, supplementary to this composition as its details, a multitude of spots and shapes for taste to judge.

Hence, taste does not indiscriminately check all intuitions produced by the imagination. Rather, taste starts by selecting a general outline and proceeds further and further into the details. The artist does not begin with a mental image of all possible wing profiles, claw shapes, and so on, which

33 Maurice Merleau-Ponty, 'Indirect Language and the Voices of Silence', in *Signs*, trans. Richard McCleary (Evanston, IL: Northwestern University Press, 1964), 45.

would be overwhelming. Rather, his or her freely original imagination interacts with taste at different levels and thereby obtains free lawfulness. In the end the artist may need to revise a choice made by taste at an earlier stage, which was entirely appropriate then but inconsistent with the details added later. Artists often modify their drafts.

Analogously, the process of formulating a complex proof requires many subsidiary steps, which are not produced and judged in one go. When the imagination generates a multitude of possibilities for the first step, the power of judgement cannot immediately identify the correct move according to the theorem or theory, for we are still trying to access the latter and do not yet have insight into all subsequent steps that would lead to it. Instead, just as in artistic creation, the power of judgement remains aesthetic and singles out a subsidiary step which appears most suitable for constructing some as-yet indeterminate concept. Put differently, without knowing whether and how exactly this step would lead to the theorem or law, we judge by the mere feeling of lawfulness without law, by means of which we eventually approach the law. During this procedure, the imagination freely presents subsidiary steps and freely conforms to the understanding's lawfulness in general. In its originality, the imagination is free from conceptual guidance, and yet, following the aesthetic guidance, the imagination also displays lawfulness that is free from conceptual accordance. Therefore, the proof is artistic in its formulation. We select an initial, seemingly appropriate step with taste, and our imagination further generates a multitude of possibilities for a subsequent step, which we evaluate with taste once again, such that the two faculties interact progressively.[34]

In the earlier example, the drawing of the subsidiary lines OA, OB, and OC does not immediately prove that $\angle C$ is a constant. Instead, the exact serviceability of these lines remains indeterminate, such that the whole composition is only judged to be rich in suggestions, or rather, 'beautiful'. Further, we avail ourselves of the properties of isosceles triangles (that is, two

[34] Wenzel makes a similar observation: 'In the process of trying out new things, a mathematician is like a painter.' Christian Helmut Wenzel, 'The Art of Doing Mathematics', in *Creativity and Philosophy*, ed. Berys Gaut and Matthew Kieran (Abingdon: Routledge, 2018), 325. But Wenzel does not spell out the *multiple* and *progressive* interactions between the imagination and the aesthetic power of judgement. More importantly, Wenzel identifies the imagination's freedom in an aesthetic judgement with 'trying out new things, new objects and new methods'; and so, he does not clearly distinguish the imagination's free originality (in trying out new things) from its free lawfulness (in the experience of beauty). See his 'Art and Imagination in Mathematics', in *Imagination in Kant's Critical Philosophy*, ed. Michael L. Thompson (Berlin: de Gruyter, 2013), 63.

angles opposite the legs must be equal) and of triangles in general (that is, interior angels add to a constant) to complete the demonstration. For another example, Figure 3 below illustrates the initial step in Hippasus' proof of the irrationality of √2. The very discovery of the unforeseen relation between the number and the geometrical shapes displays not only free creativity but also free lawfulness, as it indicates (without specifying) some as-yet indeterminate conceptual unity and thereby brings about a beautiful feeling. Based on these two squares, we construct smaller and even smaller squares (as shown in Figure 4) until we conclude that there is no 'common unit of measure' for the side length and the diagonal length in any square, that is, their ratio must be irrational.

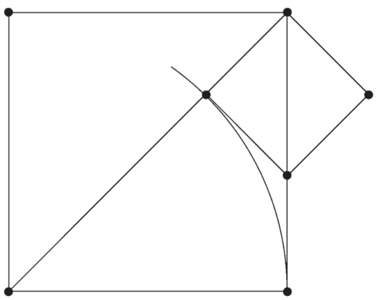

Fig. 3

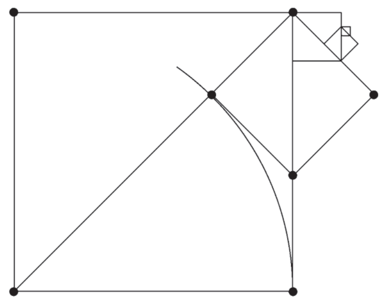

Fig. 4

Conversely to artistic creation, however, the ultimate goal of a mathematical proof is to construct a conceptual unity of such steps, otherwise we would have to retreat, revise, and restart. We would discard a step if it did not follow strictly logically from the assumptions made and the axioms of the mathematical theory in question. What is eventually proven, in each step, must be true and useful. Mathematicians are rarely so lucky to succeed in one try. Rather, they experiment back and forth, guided by the feeling of beauty, namely, by the pleasure in the imagination's free lawfulness.[35] Meanwhile, the imagination's original creativity (which constitutes genius) can be entirely free from guidance.

Some commentators underscore the role of the imagination's creativity in the experience of mathematical beauty. Angela Breitenbach contends that 'subsidiary steps' in a demonstration derive neither analytically from concepts nor empirically from given sensory data. Instead, it is the 'free play' of our

[35] One might contend that the mathematical experimentation is guided by the 'indeterminate concept', while the beautiful feeling is the result of finding the right move. But in my view, it is exactly through the feeling that we judge an intermediate move to be 'right', that is, corresponding to some indeterminate concept.

imagination that offers 'different ways of combining the sensory fold' and 'produces sensible unities'. And so, judgements about the beauty of mathematical proofs are 'grounded in the creative process of the imagination'.[36]

While Breitenbach helpfully identifies the connection between mathematical proofs and aesthetic creativity, her approach does not clearly distinguish between the free originality and the free lawfulness of the imagination. As I have shown, it is one thing that the imagination's original output remains 'undetermined by further conceptual rules', that is, free from certain conceptual guidance, but it is quite another that there is 'no further rule that can account for the harmony' and that the imagination's lawfulness is free from certain conceptual accordance. In the former, there is no concept to explain the possibility of the imagination's lawfulness, even if the lawfulness turns out to fit a concept. In the latter, there is no concept to explain the imagination's lawfulness itself, exactly because it does not correspond to any concept.

Breitenbach states that the satisfaction in beauty arises as a result of 'an unexpected agreement between our imaginative play [...] and the conceptual insight gained thereby'.[37] But as I see it, the unexpectedness, namely, the freedom in the imagination's original production, does not distinguish beauty from perfection, for a satisfaction in perfection can be unexpected as well. Moreover, the so-called 'conceptual insight' straightforwardly contradicts the 'lawfulness without law' that is requisite for beauty.

A complete demonstration can certainly bring about satisfaction, especially when it shows simplicity, hits on the essence of things, and provides more understanding for extensive applications. Nevertheless, following Kant's dichotomy, we should ascribe such a satisfaction to the perfection of things.[38] By contrast, a mathematician experiences a beautiful feeling during the process of formulating a demonstration, that is, in the sensory manifolds that he or she experiments with and his or her taste judges to be compatible with some indeterminate concept.[39] Moreover, an apprentice, who studies

36 Angela Breitenbach, 'Beauty in Proofs: Kant on Aesthetics in Mathematics', *European Journal of Philosophy* 23 (2015): 966–69. For similar accounts, see Donald W. Crawford, 'Kant's Theory of Creative Imagination', in *Essays in Kant's Aesthetics*, ed. Ted Cohen and Paul Guyer (Chicago: University of Chicago Press, 1982), 166, and Wenzel, 'Art and Imagination', 57, 65.

37 Breitenbach, 'Beauty in Proofs', 968.

38 From a Kantian perspective, I would dismiss Cellucci's account that a demonstration can be beautiful in terms of 'providing understanding' and showing 'why' ('Mathematical Beauty', 348–49).

39 It often happens that the process of discovering a proof is complex (with many steps tried, examined, and rejected), while the actual proof turns out to be simple and straightforward. As Wenzel observes: 'the whole process is visible from the inside, whereas in third-person perspective, from the outside, this is hardly possible'

the demonstration, also feels beauty in following each step shown to him or her. While his or her imagination is bound to determinate steps and thus not original, it nevertheless acquires free lawfulness in representing these steps. The difference is analogous to that between the creation and the appreciation of a beautiful artwork. Here, too, we see the importance in distinguishing between the two respects of Kant's notion of 'productive imagination'. Breitenbach's approach, insufficient in this regard, cannot explain the beautiful experience of the process of studying a demonstration.

My account of artistic proofs also facilitates a transition from subjective purposiveness to objective purposiveness in Kant's third *Critique*, a transition Kant himself fails to address. For Kant, we call something 'purposive' insofar as we must assume its foundation to be 'a will that has arranged it so in accordance with the representation of a certain rule' (*CJ*, AA 5:220). The aesthetic power of judgement represents subjective purposiveness through the mere feeling of pleasure. In judging a beautiful form, our imagination freely harmonizes with the understanding, which we can only explain by assuming the form's accordance with some indeterminate concept. Subjective (that is, completely non-conceptual) purposiveness must be formal, for we hereby appeal to the mere form of purposiveness (that is, a purposive causality), abstracting from determinate concepts of purpose.

By contrast, the logical power of judgement represents objective purposiveness according to concepts rather than mere feelings. In most cases, objective purposiveness is also material rather than formal. For instance, when we find a regular hexagon in the sand in an apparently uninhabited land, we consider it to be purposive insofar as we can only explain its possibility by assuming some creation according to the concept of 'hexagon' (*CJ*, AA 5:370).

Meanwhile, in § 62, Kant declares the 'intellectual purposiveness' of the useful properties of shapes or numbers in mathematics to be 'merely formal (not real), i.e., as purposiveness that is not grounded in a purpose, for which teleology would be necessary, but only in general', although it is 'objective (not, like the aesthetic, subjective)' (*CJ*, AA 5:364). A circle's property enables us to draw equivalent angles, for which it is objectively purposive. Nevertheless, insofar as the property also serves to construct infinitely many types of shapes, we ground its possibility in a concept 'only in general', such that its purposiveness must be formal, that is, not bounded to any particular, determinate concept.

('Beauty, Genius, and Mathematics', 326). In my view, we may find beauty during the 'examination' of possible steps, even if they are eventually rejected, and perfection in the simplicity of the actual proof.

On my reading, the 'objective, formal purposiveness' in mathematics is intended to be the mediation between the 'subjective, formal purposiveness' in aesthetics and the 'objective, material purposiveness' in teleology. That said, Kant maintains a dichotomy between subjective and objective purposiveness. This dichotomy, though necessary and convincing, keeps beauty and perfection strictly apart and leaves us in an absolute either/or situation.

In my account, an artistic proof displays beauty in its process but perfection in its completion. And so, in the very same proof we represent both subjective and objective purposiveness. It follows that we may regard beauty as indicative of some possible law that awaits exploration. For instance, we may compose beautiful music much the same way as we construct the isosceles triangles in the circle example, and the music may lead to a determinate arithmetic rule much the same way as the triangles lead to the Law of Sines. Kant suggests we consider music as 'the beautiful play of sensations (through hearing)', and yet to think 'mathematically' about 'the proportion of the oscillations in music and of the judging of them' (*CJ*, AA 5:325). After all, our mind plays in music with 'properties of numbers' (*CJ*, AA 5:363). Kant also takes notice of Pythagoras' discovery 'of the numerical relation among the tones, and of the law by which they alone produce a music'.[40] Indeed, the search for mathematical proofs is analogous to the creation of beautiful artworks.

For another instance, Figure 5 below shows the spiral form that we observe in many conches, flowers, and even typhoons. Hypothetically, this form may correspond to the Fibonacci Sequence (1, 1, 2, 3, 5, 8, 13, 21…). Figure 6 shows how we prove this correspondence by introducing subsidiary squares that relate the spiral form to the mathematical principle. On the one hand, the form, in its own terms, arouses a purely aesthetic satisfaction. On the other hand, the eventual numerical, conceptual unity, encompassing both the form and the subsidiary shapes, brings about an intellectual satisfaction. And the beautiful construction of many squares, as intermediate steps in this demonstration, reconciles the gap between the beautiful form and the conceptual unity.

[40] Immanuel Kant, *Theoretical Philosophy after 1781*, trans. Gary Hatfield et al. (Cambridge: Cambridge University Press, 2004), AA 8:392. As Giordanetti points out, while the numerical ratios result in musical harmony, they do not display some sort of 'objective formal beauty' but rather 'correspond' to 'the play of the cognitive faculties'. Piero Giordanetti, 'Objektive Zweckmäßigkeit, objektive und formale Zweckmäßigkeit, relative Zweckmäßigkeit (§§ 61–63)', in *Immanuel Kant: Kritik der Urteilskraft*, ed. Otfried Höffe (Berlin: Akademie, 2008), 220.

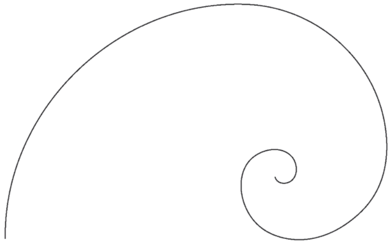

Fig. 5

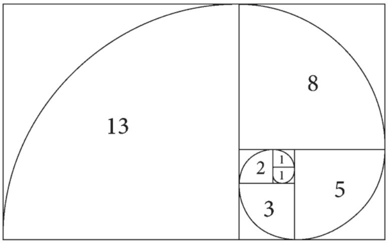

Fig. 6

Let me conclude this paper with a metaphor. We are explorers seeking a mountaintop on a foggy day with limited vision. Even if we know what the top should look like, the way to it is shrouded in mystery. Before us are many paths leading to somewhere unknown, so we have to experiment with a path that is the most ascendant and suitable for accessing some higher place only in general. Whenever we encounter a fork, we just deploy the same strategy and move on. Suppose we arrive at an impasse or find that all further paths descend, we can always return, revise the route, and retry. The mere feeling of ascendancy, without any determinate concept of where the path leads to, guides our exploration; this is the beautiful feeling that guides an artistic proof. The beauty of a proof is exactly the beauty which mathematicians feel in constructing and apprentices feels in studying a proof. If we eventually reach the top, we celebrate our achievement and admire the simplicity and convenience of a shortcut – much like the satisfaction in the perfection of a completed proof and its steps. On the other hand, it could happen that, instead of reaching the top or acquiring a proof, we encounter a set of Penrose stairs and linger in the mere pleasure of climbing up, much as artists and their audience linger with a beautiful form without ever ascertaining a concept thereof.

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